Complex Numbers 2: Sums of Series and nth Roots

For complex numbers w, z

$$\sum_{r=0}^{n-1} wz^r = \frac{w(z^n - 1)}{z - 1} \qquad \sum_{r=0}^{\infty} wz^r = \frac{w}{1 - z}$$

Example

Given that $z = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$ where n is a positive integer, show that

$$1 + z + z^2 + \cdots + z^{n-1} = 1 + i\cot\left(\frac{\pi}{2n}\right)$$

Example

$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{8i\theta}$

a. Show that $S = \frac{e^{\frac{9i\theta}{2}} \sin 4\theta}{\sin\frac{\theta}{2}}$

Let $P = \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos 8\theta$
And $Q = \sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin 8\theta$

b. Use your answer to part a. to show that $P = \cos\frac{9\theta}{2} \sin 4\theta \csc\frac{\theta}{2}$ and find similar expressions for Q and $\frac{P}{Q}$

If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct roots.

Example

a. Solve the equation $z^3 = 1$

b. Represent your solutions to part a. on an Argand diagram

c. Show that the three cube roots of 1 can be written as $1, \omega$ and ω^2 where $1 + \omega + \omega^2 = 0$

If $z^n = 1$ then $z = e^{\frac{2\pi i k}{n}}$ because $e^{2\pi i k} = 1$

The roots are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$

In an Argand diagram the roots form a regular *n* - agon

The sum of the roots is zero.

If $z^n = s$ and one root is z_1 the other roots are $z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$

In an Argand diagram these roots form a regular *n* – agon too.

Example

Solve the equation $z^4 = 2 + 2i\sqrt{3}$

Example

Solve the equation $z^3 + 4\sqrt{2} + 4\mathrm{i}\sqrt{2} = 0$

Example

The point ($\sqrt{3}$, 1) lies at one vertex of an equilateral triangle.

The centre of the triangle is at the origin.

a. Find the coordinates of the other vertices of the triangle.

b. Find the area of the triangle.