# Modelling with Differential Equations Ch8.3: Damped and Forced SHM

Damped Harmonic Motion $$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

If $k^2 > 4\omega^2$ this is HEAVY DAMPING

If $k^2 = 4\omega^2$ this is CRITICAL DAMPING

If $k^2 < 4\omega^2$ this is LIGHT DAMPING

Forced Harmonic Motion $$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

A particle $P$ of mass 0.5 kg moves in a horizontal straight line.

At time $t$ seconds, the displacement of $P$ from a fixed point $O$, on the line is $x$ m and the velocity of $P$ is $v$ $ms^{-1}$.

A force of magnitude $8x$ N acts on $P$ in the direction $PO$.

The particle is also subject to a resistance of magnitude $4v$ N.

When $t = 0$, $x = 1.5$ and $P$ is moving in the direction of $x$ increasing with speed 4 $ms^{-1}$.

a. Show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0$

b. Find the value of $x$ when $t = 1$.

A particle *P* hangs freely in equilibrium attached to one end of a light elastic string.

The other end of the string is attached to a fixed point *A*.

The particle is now pulled down and held at rest in a container of liquid which exerts a resistance to motion on *P*.

*P* is then released from rest.

While the string remains taut and the particle in the liquid, the motion can be modelled using the equation $\frac{d^2x}{dt^2} + 6k\frac{dx}{dt} + 5k^2x = 0$.

Find the general solution to the differential equation and state the type of damping that the particle is subject to.

One end of a light elastic spring is attached to a fixed point *A*.

A particle *P* is attached to the other end and hangs in equilibrium vertically below *A*.

The particle is pulled vertically down from its equilibrium position and released from rest.

A resistance proportional to the speed of *P* acts on *P*.

The equation of motion of *P* is given as $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$.

a. Find the general solution to the differential equation.

b. Write down the period of oscillation in terms of *k*.

A particle $P$ of mass 1.5 kg is moving on the $x$ axis.

At time $t$ the displacement of $P$ from the origin $O$ is $x$ m and the speed is $v$ ms$^{-1}$.

Three forces act on $P$, namely a restoring force of magnitude $7.5x$ N, a resistance to the motion of $P$ of magnitude $6v$ N and a force of magnitude 12 sin $t$ N acting in the direction $OP$.

When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 2$.

a. Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8\sin t$

b. Find $x$ as a function of $t$

c. Describe the motion when $t$ is large.

A particle $P$ is attached to end $A$ of a light elastic string $AB$.

Initially the particle and the string lie at rest on a smooth horizontal plane.

At time $t = 0$, the end $B$ of the string is set in motion and moves with constant speed $U$ in the direction $AB$, and the displacement of $P$ from $A$ is $x$.

Air resistance acting on $P$ is proportional to its speed.

The subsequent motion can be modelled by the differential equation
$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$.

Find an expression for $x$ in terms of $U$, $k$ and $t$.