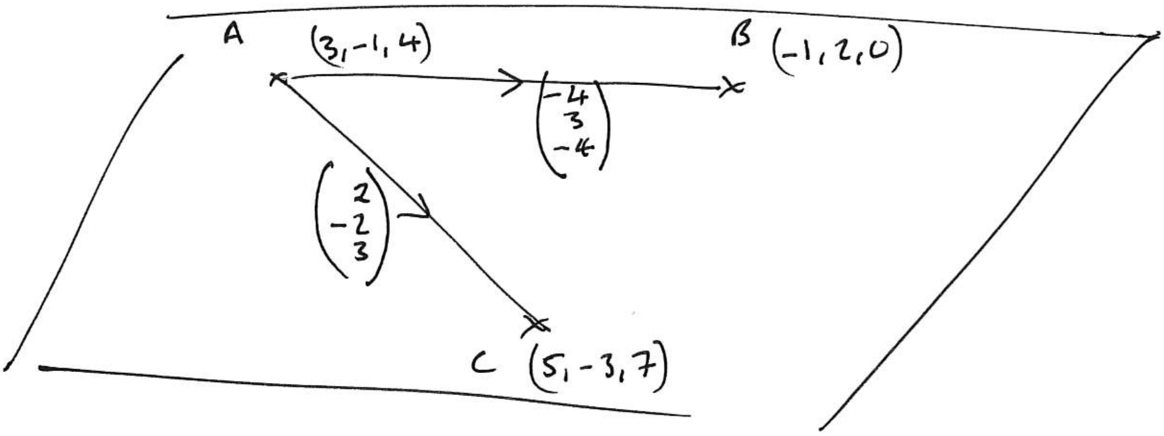
Further Pure 1 Practice Paper 2.

①



a) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} = i(9-8) - j(-12 - {}^{-}8) + k(8-6)$

$= i + 4j + 2k$

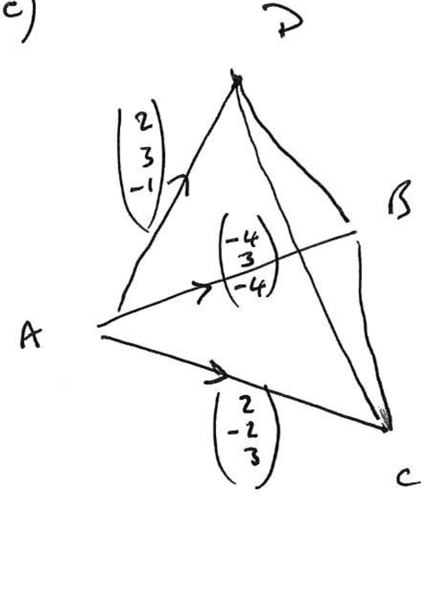
b) $\underline{n} = i + 4j + 2k$

$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} \Rightarrow \underline{r} \cdot (i + 4j + 2k) = (-i + 2j + 0k) \cdot (i + 4j + 2k)$

$= -1 + 8$

$\underline{r} \cdot (i + 4j + 2k) = 7$

c)



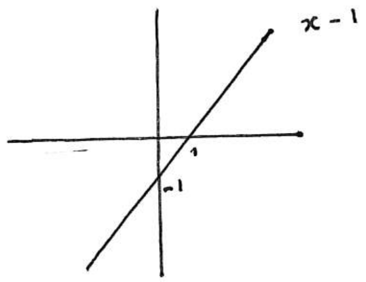
$\text{Vol} = \frac{1}{6} \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$

$= \frac{1}{6} \begin{pmatrix}2\\3\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\4\\2\end{pmatrix}$

$= \frac{1}{6} (2 + 12 - 2)$

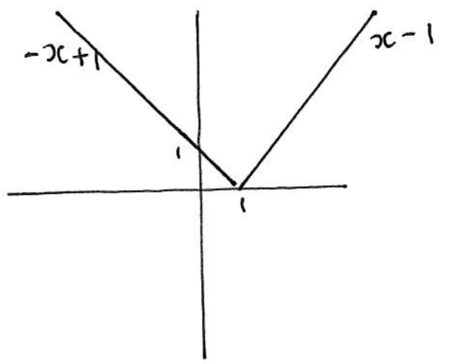
$= \underline{\underline{2}}$

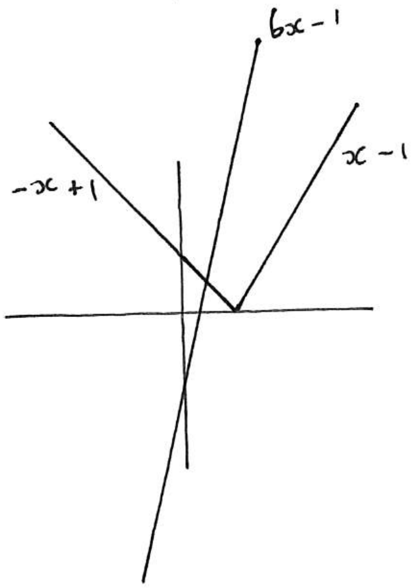
(2) $|x-1| > 6x - 1$



$y = x - 1$

$y = |x-1|$





$y = |x-1|$
and $y = 6x-1$

Intersection when $-x+1 = 6x-1$

$2 = 7x$

$x = 2/7$

$|x-1| > 6x - 1$

for $x < \frac{2}{7}$

VD

③ $y = \sec^2 x$

a let $u = \sec x \Rightarrow \dfrac{du}{dx} = \sec x \tan x$ (F.S.)

$y = u^2 \Rightarrow \dfrac{dy}{du} = 2u.$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2(\sec x) \cdot (\sec x \tan x)$$

$$= 2\sec^2 x \tan x.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\overset{u}{\left(2\sec^2 x\right)}\overset{v}{\left(\tan x\right)}$$

$$= \overset{u'}{\left(4\sec^2 x \tan x\right)}\overset{v}{\left(\tan x\right)} + \overset{u}{\left(2\sec^2 x\right)}\overset{v'}{\left(\sec^2 x\right)}$$

$$= 4\sec^2 x \tan^2 x + 2\sec^4 x$$

[since $\tan^2 x = \sec^2 x - 1$] $= 4\sec^2 x\left[\sec^2 x - 1\right] + 2\sec^4 x$

$$= 4\sec^4 x - 4\sec^2 x + 2\sec^4 x$$

$$= 6\sec^4 x - 4\sec^2 x$$

b) Taylor Series: $f(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

will need $\frac{d^3y}{dx^3}$.

$\frac{d^2y}{dx^3} = 6\sec^4 x - 4\sec^2 x$

let $u = \sec x \Rightarrow \frac{d^2y}{dx^2} = 6u^4 - 4u^2 \Rightarrow \frac{d^3y}{du^3} = 24u^3 - 8u$.

$\frac{du}{dx} = \sec x \tan x$.

$\Rightarrow \frac{d^3y}{dx^3} = 24\sec^3 x \,.\, \sec x \tan x - 8\sec x \,.\, \sec x \tan x$.

at $x = \frac{\pi}{4}$:

$y = \sec^2(\pi/4) = 2$

$\frac{dy}{dx} = 2\sec^2(\pi/4)\operatorname{Tan}(\pi/4) = 4$

$\frac{d^2y}{dx^2} = 6\sec^4\left(\frac{\pi}{4}\right) - 4\sec^2(\pi/4) = 16$

$\frac{d^3y}{dx^3} = 24\sec^4\left(\frac{\pi}{4}\right)\operatorname{Tan}\left(\frac{\pi}{4}\right) - 8\sec^2(\pi/4)\operatorname{Tan}(\pi/4) = 80$

Taylor Series

$\operatorname{Sec}(x) \simeq 2 + 4(x - \pi/4) + \frac{16}{2}(x - \pi/4)^2 + \frac{80}{6}(x - \pi/4)^3 \ldots$

(4)

a)

$x_0 = 0 \quad y_0 = 1 \quad \frac{dy}{dx_0} = 0^2 - 1^2 = -1$

$x_1 = 0.05 \quad y_1 = 1 + -1 \times 0.05 = 0.95 \quad \frac{dy}{dx_1} = 0.05^2 - 0.95^2 = -0.9$

$x_2 = 0.1 \quad y_2 = 0.95 + -0.9 \times 0.05 = 0.905 \quad \frac{dy}{dx_2} = 0.1^2 - 0.905^2 = -0.809025$

$$\boxed{y_n = y_{n-1} + \frac{dy}{dx_{n-1}} \times h}$$

b)

$$\frac{dy}{dx} = x^2 - y^2$$

$$\frac{d^2y}{dx^2} = 2x - 2y\frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = 2 - 2\frac{dy}{dx}\frac{dy}{dx} - 2y\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 2 = 0.$$

when $x = 0$

$y = 1$

$\frac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = 2(0) - 2(1)(-1) = 2$$

$$\frac{d^3y}{dx^3} = 2 - 2(-1)^2 - 2(1)(2) = 2 - 2 - 4 = -4$$

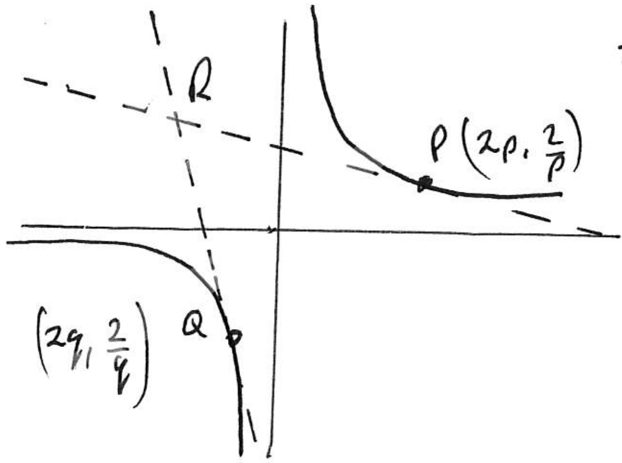
c) Series solution at $x = 0$

$$f(x) \approx \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \dots$$

$$= \frac{1}{1} + \frac{(-1)}{1}x + \frac{2}{2}x^2 + \frac{-4}{6}x^3$$

$$= 1 - x + x^2 - \frac{2}{3}x^3 \dots$$

(5)



$xy = 4$

for $\quad xy = 4 \quad \Rightarrow \quad y = 4x^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = -4x^{-2}$

$$= \frac{-4}{x^2}.$$

Tangent at P: $\quad y - \frac{2}{p} = \frac{-4}{(2p)^2}(x - 2p)$

$$y = -\frac{1}{p^2}x + \frac{2}{p} + \frac{2}{p} = -\frac{1}{p^2}x + \frac{4}{p}.$$

Tangent at Q: $\quad y = -\frac{1}{q^2}x + \frac{4}{q}.$

At R: $\quad y = y \Rightarrow -\frac{1}{p^2}x + \frac{4}{p} = -\frac{1}{q^2}x + \frac{4}{q}$

$[\times p^2q^2]$ $\quad \Rightarrow -q^2x + 4pq^2 = -p^2x + 4p^2q$

$$\Rightarrow (p^2 - q^2)x = 4pq(p - q)$$

$$\Rightarrow x = \frac{4pq\,\cancel{(p-q)}}{(p+q)\cancel{(p-q)}} \quad ✓$$

Sub x into tgt at P:

$$y = -\frac{1}{p^2}\left(\frac{4pq}{p+q}\right) + \frac{4}{p} \times \frac{(p+q)}{(p+q)}$$

$$= -\frac{4}{p}\left(\frac{q}{p+q}\right) + \frac{4}{p}\frac{(p+q)}{(p+q)}$$

$$= \frac{-\cancel{4q} + 4p + \cancel{4q}}{p(p+q)} = \frac{4}{p+q}.$$

b) Since $xy = 3$ for Point R,

$$\Rightarrow \frac{4pq}{p+q} \times \frac{4}{p+q} = 3.$$

$$\Rightarrow 16pq = 3.(p+q)^2$$

$$16pq = 3p^2 + 6pq + 3q^2$$

$$3q^2 - 10pq + 3p^2 = 0$$

Factorise or use quad formula or complete the square

$$q = 3p \quad \text{or} \quad q = \frac{1}{3}p.$$

(6) a) $b \times c = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = i(1-1) - j(-3-2) + k(3 - -2)$

$= 5j + 5k.$

b) $\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 5$

c) Area $= \frac{1}{2} \left| \overrightarrow{OB} \times \overrightarrow{OC} \right| = \frac{1}{2}\sqrt{5^2+5^2} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}.$

d) Tetrahedron $= \frac{1}{6} \overrightarrow{OA} \cdot \left(\overrightarrow{OB} \times \overrightarrow{OC} \right)$

$= \frac{1}{6} \times 5 = \frac{5}{6}.$

⑦ $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6x = 2e^{-t}$$

Aux Eqn: Solve $m^2 + 5m + 6 = 0$

$$(m+3)(m+2) = 0$$

$$m = -3 \quad \text{or} \quad -2.$$

$\Rightarrow$ Comp Fn $\quad x = Ae^{-3t} + Be^{-2t}.$

Particular Integral:

Sub $x = ke^{-t}.$ $\quad \left[\Rightarrow \frac{dx}{dt} = -ke^{-t} \quad \frac{d^2x}{dt^2} = ke^{-t}\right]$

$\Rightarrow \quad ke^{-t} - 5ke^{-t} + 6ke^{-t} = 2e^{-t}$

$\Rightarrow \quad 2k = 2 \quad \Rightarrow \quad k = 1$

Hence $\quad \underline{x = Ae^{-3t} + Be^{-2t} + e^{-t}}$

$t = 0, \quad x = 0 \quad \Rightarrow \quad 0 = A + B + 1 \quad (1)$

$t = 0 \quad \frac{dx}{dt} = 2 \quad \& \quad \frac{dx}{dt} = -3Ae^{-3t} - 2Be^{-2t} - e^{-t}.$

$\Rightarrow \quad 2 = -3A - 2B - 1$

$\Rightarrow \quad 3A + 2B = -3 \quad (2)$

Solve Sim Equs 1 & 2

$\Rightarrow \quad A = -1 \qquad B = 0$

$\Rightarrow \quad x = -e^{-3t} + e^{-t}$ 3

b) Solve $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 3e^{-3t} - e^{-t}$

$\Rightarrow \quad e^{-t}\left(3e^{-2t} - 1\right) = 0$

Since $e^{-t} \neq 0 \qquad 3e^{-2t} = 1 \Rightarrow e^{-2t} = \frac{1}{3}$

$\Rightarrow -2t = \ln\left(\frac{1}{3}\right)$

$-2t = -\ln(3)$

$t = \frac{1}{2}\ln(3)$

Sub t into 3 to find x:

$x = -e^{-\frac{3}{2}\ln(3)} + e^{-\frac{1}{2}\ln(3)}$

$= -\left(e^{\ln 3}\right)^{-3/2} + \left(e^{\ln 3}\right)^{-\frac{1}{2}} = -3^{-3/2} + 3^{-\frac{1}{2}}$

$= \frac{-1}{3^{3/2}} + \frac{1}{3^{1/2}} = \frac{-1}{3^{3/2}} + \frac{3}{3^{3/2}} = \frac{2}{3^{3/2}}$

$= \frac{2\sqrt{3}}{3^2} = \frac{2\sqrt{3}}{9}.$

Max Occurs when $\frac{d^2x}{dt^2} < 0$.

at $t = \frac{1}{2}\ln 3$

$$\frac{d^2x}{dt^2} = 2e^{-t} - 6x - 5\frac{dx}{dt}.$$

$$= 2e^{-\frac{1}{2}\ln 3} - 6 \cdot \frac{2\sqrt{3}}{9} - 5 \times 0.$$

$$= 2\left(e^{\ln 3}\right)^{-\frac{1}{2}} - \frac{4\sqrt{3}}{3}$$

$$= 2 \cdot \frac{1}{\sqrt{3}} - \frac{4\sqrt{3}}{3}$$

$$= \frac{2\sqrt{3}}{3} - \frac{4\sqrt{3}}{3} < 0.$$

Hence Max.