L6 HL Differentiation from First Principles

1 For the function $f(x) = x^2$, use the definition of the derivative to show that:

a $f'(2) = 4$ b $f'(-3) = -6$ c $f'(0) = 0$ d $f'(50) = 100$

2 $f(x) = x^2$

a Show that $f'(x) = \lim_{h \to 0} (2x + h)$. b Hence deduce that $f'(x) = 2x$.

3 The point A with coordinates $(-2, -8)$ lies on the curve with equation $y = x^3$. At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0} (12 - 6h + h^2)$. b Deduce the value of g.

(P) 4 The point A with coordinates $(-1, 4)$ lies on the curve with equation $y = x^3 - 5x$.
The point B also lies on the curve and has x-coordinate $(-1 + h)$.

a Show that the gradient of the line segment AB is given by $h^2 - 3h - 2$.

b Deduce the gradient of the curve at point A.

Problem-solving

Draw a sketch showing points A and B and the chord between them.

(E/P) 5 Prove, from first principles, that the derivative of $6x$ is 6. **(3 marks)**

(E/P) 6 Prove, from first principles, that the derivative of $4x^2$ is $8x$. **(4 marks)**

(E/P) 7 $f(x) = ax^2$, where a is a constant. Prove, from first principles, that $f'(x) = 2ax$. **(4 marks)**

1 a $f(x) = x^2$

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(2+h)^2-2^2}{h} = \lim_{h\to 0}\frac{4+4h+h^2-4}{h} = \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0}\frac{h(4+h)}{h} = \lim_{h\to 0}(4+h)$$

As $h \to 0$, $4 + h \to 4$.
So $f'(2) = 4$

b $$f'(-3) = \lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} = \lim_{h\to 0}\frac{(-3+h)^2-(-3)^2}{h} = \lim_{h\to 0}\frac{9-6h+h^2-9}{h} = \lim_{h\to 0}\frac{-6h+h^2}{h} = \lim_{h\to 0}\frac{h(-6+h)}{h} = \lim_{h\to 0}(-6+h)$$

As $h \to 0$, $-6 + h \to -6$.
So $f'(-3) = -6$

c $$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(0+h)^2-0^2}{h} = \lim_{h\to 0}\frac{h^2}{h} = \lim_{h\to 0} h$$

$f'(0) = 0$

d $$f'(50) = \lim_{h\to 0}\frac{f(50+h)-f(50)}{h} = \lim_{h\to 0}\frac{(50+h)^2-50^2}{h} = \lim_{h\to 0}\frac{2500+100h+h^2-2500}{h} = \lim_{h\to 0}\frac{100h+h^2}{h} = \lim_{h\to 0}\frac{h(100+h)}{h} = \lim_{h\to 0}(100+h)$$

As $h \to 0$, $100 + h \to 100$.
So $f'(50) = 100$

2 a $f(x) = x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h)$$

b As $h \to 0$, $2x + h \to 2x$.
So $f'(x) = 2x$

3 a $y = x^3$, therefore $f(x) = x^3$

$$g = \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} = \lim_{h\to 0}\frac{(-2+h)^3-(-2)^3}{h} = \lim_{h\to 0}\frac{-8+3(-2)^2h+3(-2)h^2+h^3+8}{h} = \lim_{h\to 0}\frac{12h-6h^2+h^3}{h} = \lim_{h\to 0}\frac{h(12-6h+h^2)}{h} = \lim_{h\to 0}(12-6h+h^2)$$

3 b As $h \to 0$, $12 - 6h + h^2 \to 12$.
So $g = 12$

4 a y-coordinate of point B
$= (-1 + h)^3 - 5(-1 + h)$
Gradient of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(-1+h)^3 - 5(-1+h) - 4}{(-1+h) - (-1)}$$

$$= \frac{-1 + 3h - 3h^2 + h^3 + 5 - 5h - 4}{h}$$

$$= \frac{h^3 - 3h^2 - 2h}{h}$$

$= h^2 - 3h - 2$

b At point A, as $h \to 0$, $h^2 - 3h - 2 \to -2$.
So gradient $= -2$

5 $f(x) = 6x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{6(x+h) - 6x}{h}$$

$$= \lim_{h \to 0} \frac{6x + 6h - 6x}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h}$$

$$= \lim_{h \to 0} 6$$

So $f'(x) = 6$

6 $f(x) = 4x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(8x + 4h)}{h}$$

$$= \lim_{h \to 0} (8x + 4h)$$

As $h \to 0$, $8x + 4h \to 8x$.
So $f'(x) = 8x$

7 $f(x) = ax^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 - ax^2}{h}$$

$$= \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 - ax^2}{h}$$

$$= \lim_{h \to 0} \frac{2axh + ah^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2ax + ah)}{h}$$

$$= \lim_{h \to 0} (2ax + ah)$$

As $h \to 0$, $2ax + ah \to 2ax$.
So $f'(x) = 2ax$

Challenge

a $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{-1}{x^2 + hx}$$

b As $h \to 0$, $\frac{-1}{x^2 + hx} \to -\frac{1}{x^2}$.

So $f'(x) = -\frac{1}{x^2}$